Operational Amplifiers and Transfer Functions

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Based on notes from Dr. Mark Lawford

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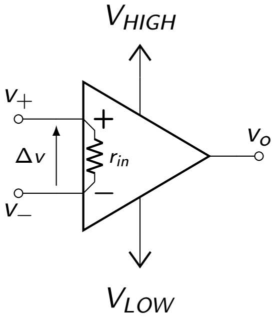
Basic Model of an Op Amp



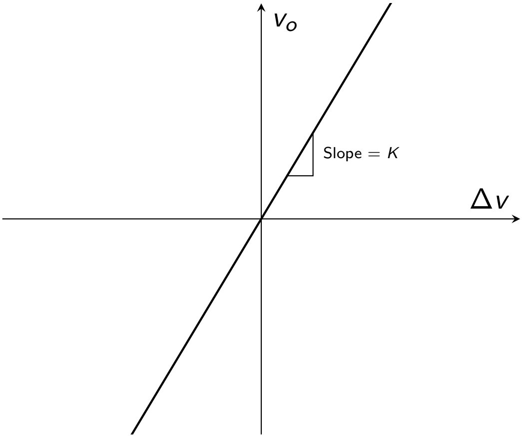
Terminals

- Two inputs:
  Non-inverting input v_+
  Inverting input v_-
- One output: v_o
- Two voltage supplies: V_{HIGH} and V_{LOW}

Characteristics

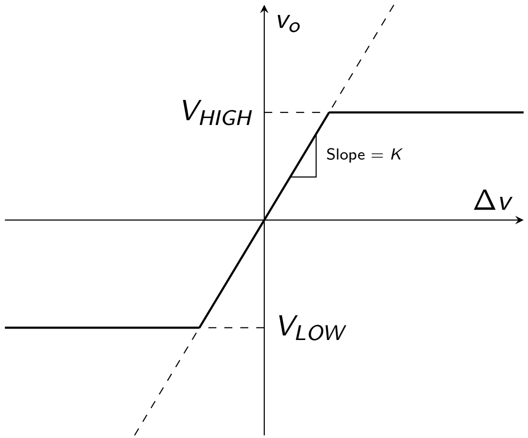
- The op-amp is a linear amplifier on the difference $\Delta v = v_+ - v_-$, so $v_o = K\Delta v$
- The gain K is very high
- The input resistance is very high ($r_{in} > 1\,\mathrm{M\Omega}$)

Ideal Linear Amplification



The output voltage v_o is linearly proportional to $\Delta v = v_+ - v_-$

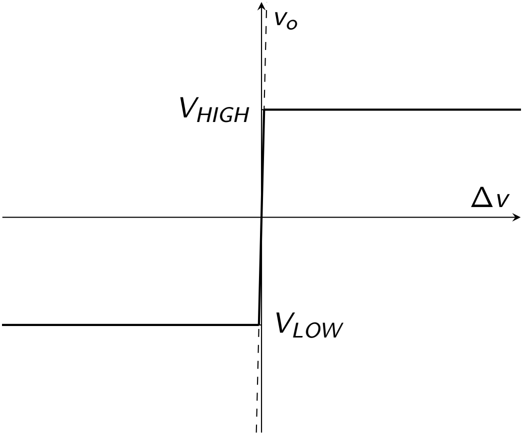
Practical Linear Amplification



In practice, the output voltage is limited by the voltages supplied to the op-amp, so v_o cannot rise above V_{HIGH} or fall below V_{LOW}.

Practical Linear Amplification

The previous picture gives a poor sense of scale for the v_o vs Δv relationship. Since the gain K is actually very large, a more accurate graph would look like this:



Remarks

- The **Linear Region** is the range from $\frac{V_{LOW}}{K}$ to $\frac{V_{HIGH}}{K}$.
- The region where the voltage is limited by the supplies is called the **Saturation Region**

Examples

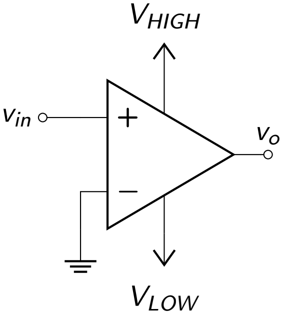
Suppose $V_{LOW} = -15\,\mathrm{V}$, $V_{HIGH} = 15\,\mathrm{V}$, $K = 100\,000$.
Therefore, the linear region is between $-0.15\,\mathrm{mV}$ and $0.15\,\mathrm{mV}$.

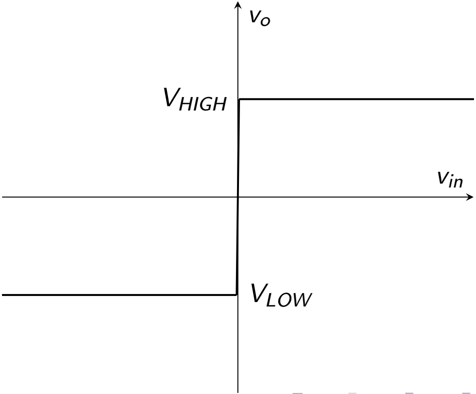
Δv	v_o	Region
$0.01\,\mathrm{mV}$	$1\,\mathrm{V}$	Linear
$0.02\,\mathrm{mV}$	$2\,\mathrm{V}$	Linear
$0.5\,\mathrm{V}$	$15\,\mathrm{V}$	Saturation
$-7\,\mathrm{V}$	$-15\,\mathrm{V}$	Saturation

Use Case 1: Comparator

A comparator is a device that outputs HIGH when the input voltage is positive and LOW when the input is negative.

To construct this device with an op-amp, connect v_- to ground (i.e. to 0 V) and connect the input signal to v_+. Now $\Delta v = v_{in} - 0$.

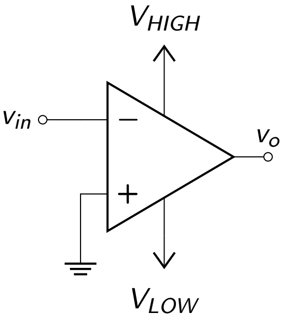


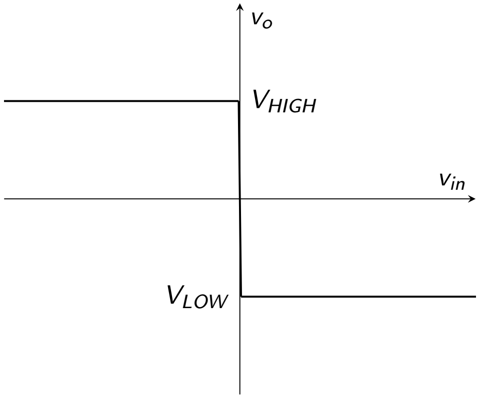


Use Case 2: Inverting Comparator

Identical to a regular comparator except when a positive v_{in} should produce a low voltage.

Connect v_+ to ground and connect the input signal to the inverting input v_-. Now $\Delta v = -v_{in}$.

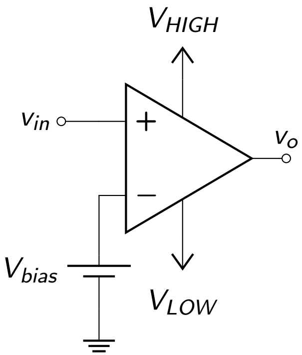


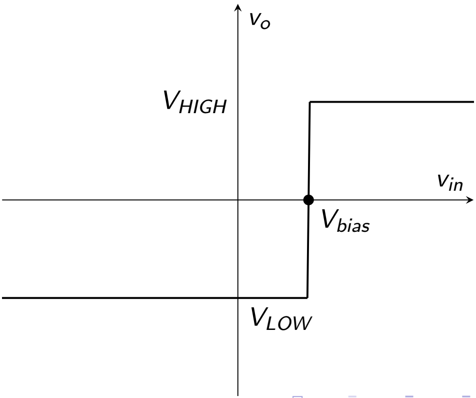


Use Case 3: Comparator with offset

Suppose you want to compare a signal against a non-zero threshold (for example, a threshold of 2.5 V could be used to determine if a 5 V digital signal is 1 or 0).

Connect your signal to v_+ and bias to v_- to set the threshold. Now $\Delta v = v_{in} - V_{bias}$.





Comparator Remarks

- Comparators are said to operate in their saturation region since the desired behaviour involves saturating the output.
- For input voltages very close to the threshold, the op-amp will be in its linear region, so the output will not be V_{HIGH} or V_{LOW} as expected.

Op-Amps with Feedback

So far, all of the circuits have been an "open-loop" configuration, which means that the inputs are independent of the output.

When feedback is used, the output is connected (possibly through components like resistors or capacitors) back to one of the inputs. This greatly expands the functionality of the op-amp.

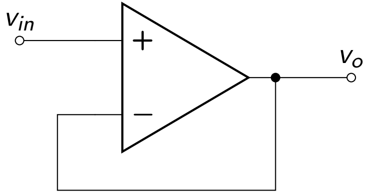
In the following circuits, we will assume that the op-amp is operating in its linear region, although it would be possible to design these circuits such that the output saturates for some inputs.

Because of this assumption, the V_{HIGH} and V_{LOW} supply voltages have been omitted from further circuits diagrams, although these connections would still be necessary for a physical circuit.

Voltage Follower

The simplest op-amp feedback circuit is a voltage follower, which is a device whose output matches its input.

To create this device, connect the input signal to v_+ and connect the output back to v_-.



Voltage Follower

To see why this is a voltage follower, use the op-amp amplification equation and the fact that v_- is tied to v_o.

$$\begin{aligned} v_o &= K\Delta v \\ v_o &= K\left(v_+ - v_-\right) \\ v_o &= K\left(v_{in} - v_o\right) \\ v_o + Kv_o &= Kv_{in} \\ v_o &= \frac{K}{K+1} v_{in} \end{aligned}$$

Since K is very large, $\frac{K}{K+1} \approx 1$ so $v_o = v_i$ and thus the output voltage matches the input voltage.

Positive and Negative Feedback

The choice to connect v_o to the inverting input v_- seems arbitrary. We could have connected it to v_+, then similar math would show that $v_o = \frac{K}{K-1} v_{in} \approx v_{in}$, which also appears to be a voltage follower.

Consider the stability of these two configurations. Suppose a small deviation causes v_o to increase slightly above v_{in}.

- If v_o is connected to v_+ and v_{in} to v_-, the deviation will make Δv positive. This change will be amplified at v_o, causing v_o to increase further, leading to runaway positive feedback until $v_o = V_{HIGH}$.
- If v_o is connected to v_- and v_{in} to v_+, the positive deviation in v_o will make Δv negative, so the amplification will induce a change in v_o that is opposite to the initial positive deviation. This negative feedback will settle at $v_o = \frac{K}{K+1} v_{in} \approx v_{in}$

Positive and Negative Feedback

Therefore, a positive-feedback voltage follower configuration is impractical since is succumbs to runaway feedback.

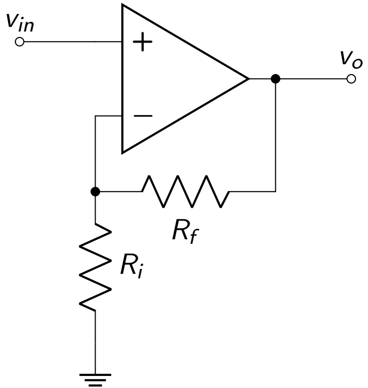
A negative-feedback voltage follower is stable and will settle when $v_o = v_{in}$.

We can now create some more interesting op-amp circuits using feedback.

Amplifier

The gain K of the op-amp is too large for most amplification purposes, and it is susceptible to changes in temperature, humidity, and other degradation.

Consider the following op-amp circuit using a resistor circuit to connect v_o to v_-.



Amplifier

To analyze this circuit, use the fact that the internal resistance of the op-amp r_{in} is very large (this resistor is not shown, see the second slide). This means that there is effectively no current flowing into either v_+ or v_-.

Combining this fact about input current with KCL, we find that

$$v_- = \frac{R_i}{R_i + R_f} v_o$$

Since the feedback is provided to the negative terminal, this system will be stable, and we can solve for the v_o vs v_{in} relationship as we did for the voltage follower.

$$
\begin{aligned}
v_o &= K\Delta v \\
v_o &= K\left(v_{in} - \frac{R_i}{R_i + R_f} v_o\right) \\
v_o\left(1 + K\frac{R_i}{R_i + R_f}\right) &= K v_{in} \\
v_o &= \frac{K}{1 + K\frac{R_i}{R_i+R_f}} v_{in} \\
v_o &= \frac{1}{\frac{1}{K} + \frac{R_i}{R_i+R_f}} v_{in}
\end{aligned}
$$

Since $\frac{1}{K} \approx 0$

$$
v_o = \frac{1}{0 + \frac{R_i}{R_i+R_f}} v_{in} = \frac{R_i + R_f}{R_i} v_{in} = \left(1 + \frac{R_f}{R_i}\right) v_{in}
$$

Amplifier

Therefore, this circuit produces an input-output relation of

$$v_o = \left(1 + \frac{R_f}{R_i}\right) v_{in}$$

which is a amplifier with a voltage gain of $\left(1 + \frac{R_f}{R_i}\right)$. Since R_i and R_f are external components, we are free to choose them as desired to set the gain of this amplifier.

Most importantly, *the gain of this amplifier circuit is independent of the gain K of the op-amp component*.

Amplifier

Refer to the amplifier circuit shown previously.

Example

What is the voltage gain of an amplifier with $R_f = 10\,\mathrm{k\Omega}$ and $R_i = 10\,\mathrm{k\Omega}$?

$$\text{gain} = 1 + \frac{R_f}{R_i} = 1 + \frac{10\,\mathrm{k\Omega}}{10\,\mathrm{k\Omega}} = 2$$

Example

If R_i is fixed at $47\,\mathrm{k\Omega}$, what value should R_f have to create a voltage gain of 20?

$$\text{gain} = 1 + \frac{R_f}{R_i} \implies 20 = 1 + \frac{R_f}{47\,\mathrm{k\Omega}} \implies R_f = 893\,\mathrm{k\Omega}$$

Further Analysis of the Amplifier Circuit

What is the value of v_- in the amplifier circuit?

Recall the relation $v_- = \frac{R_i}{R_i+R_f} v_o$ and use the equation for v_o from v_{in}.

$$
\begin{aligned}
v_- &= \frac{R_i}{R_i + R_f} v_o \\
v_- &= \frac{R_i}{R_i + R_f} \left(1 + \frac{R_f}{R_i}\right) v_{in} \\
v_- &= \frac{R_i}{R_i + R_f} \left(\frac{R_i + R_f}{R_i}\right) v_{in} \\
v_- &= v_{in}
\end{aligned}
$$

Therefore, the circuit "settles" in such a way that the voltage at v_- matches the voltage at $v_{in} = v_+$.

This result that $v_- = v_+$ seems to be a contradiction, since then $\Delta v = 0$ and thus $v_o = K\Delta v = 0$, contrary to our derivation showing that this circuit is an amplifier.

However, this equality arises because of our approximation of $\frac{1}{K} = 0$ when deriving the amplifier voltage gain. Watch what happens when this approximation is not performed.

Recall the v_- vs v_o relation and the equation just before the approximation was made:

$$v_- = \frac{R_i}{R_i + R_f} v_o \qquad v_o = \frac{1}{\frac{1}{K} + \frac{R_i}{R_i+R_f}} v_{in}$$

Combining these gives:

$$v_- = \left(\frac{R_i}{R_i + R_f}\right)\left(\frac{1}{\frac{1}{K} + \frac{R_i}{R_i+R_f}}\right) v_{in}$$

$$v_- = \frac{R_i}{\frac{R_i+R_f}{K} + R_i} v_{in}$$

$$v_- = \frac{1}{\frac{1+\frac{R_f}{R_i}}{K} + 1} v_{in}$$

$$v_- = \frac{1}{\frac{\text{gain}}{K} + 1} v_{in}$$

Therefore, we find that v_- is not exactly equal to $v_+ = v_{in}$, but is rather slightly scaled down since $\frac{1}{\frac{\text{gain}}{K}+1} < 1$. Now consider Δv:

$$
\begin{aligned}
\Delta v &= v_+ - v_- \\
&= v_{in} - \frac{1}{\frac{\text{gain}}{K} + 1} v_{in} \\
&= \left(1 - \frac{1}{\frac{\text{gain}}{K} + 1}\right) v_{in} \\
&= \frac{\frac{\text{gain}}{K} + 1 - 1}{\frac{\text{gain}}{K} + 1} v_{in} \\
&= \frac{\text{gain}}{\text{gain} + K} v_{in}
\end{aligned}
$$

Since the voltage gain is usually much less than K, the fraction $\frac{\text{gain}}{\text{gain}+K}$ is very small, which agrees with our approximation $\Delta v \approx 0$.

Given this expression for Δv, use the op-amp amplification equation to find v_o

$$v_o = K\Delta v = K\frac{\text{gain}}{\text{gain} + K}v_{in} = \frac{K \cdot \text{gain}}{\text{gain} + K}v_{in} = \frac{\text{gain}}{\frac{\text{gain}}{K} + 1}v_{in} \approx \frac{\text{gain}}{1}v_{in}$$

And so the circuit does act like an amplifier with the expected gain, so long as the circuit gain is sufficiently less than K.

Remarks on Assumptions

Notice that we did not make any assumptions or idealizations when calculating the voltage follower or amplifier behaviours, besides the known fact that K is very large. However, in the process of this analysis, the result that $v_- \approx v_+$ appeared very naturally.

This becomes an important result that greatly simplifies negative-feedback circuit analysis:

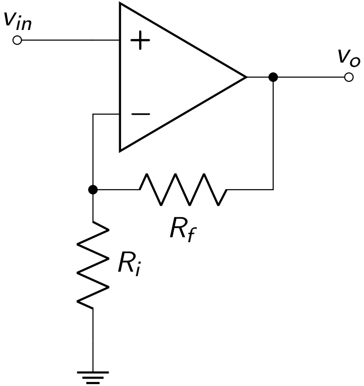
Negative Feedback Assumption

If an op-amp is configured with negative feedback, then v_o will settle at the voltage which causes v_- to equal v_+.

This assumption should not be made in circuits with positive feedback (recall the unstable voltage follower) or in circuits with no feedback (like the comparators).

Amplifier

Let's re-analyze the amplifier circuit. Since there is negative feedback, the $v_+ = v_-$ assumption can be made.



$$v_- = \frac{R_i}{R_i + R_f} v_o$$

Since $v_- = v_+$, and $v_+ = v_{in}$

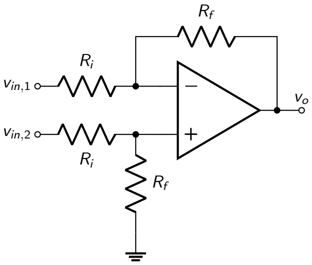
$$v_{in} = \frac{R_i}{R_i + R_f} v_o$$

$$v_o = \frac{R_i + R_f}{R_i} v_{in} = \left(1 + \frac{R_f}{R_i}\right) v_{in}$$

And so using the assumption leads to the same result as before but with *far* less calculations.

Differential Amplifier Example

Derive an equation for v_o as a function of the two inputs in this circuit:



Since the input resistance of the op-amp is high, there is no current flow into either terminal of the op-amp. Thus, KCL can be used to find:

$$v_- = \frac{R_i}{R_f + R_i} v_o + \frac{R_f}{R_f + R_i} v_{in,1}$$

$$v_+ = \frac{R_f}{R_f + R_i} v_{in,2}$$

Since there is negative feedback, we can set $v_- = v_+$, then isolate for v_o.

$$\frac{R_f}{R_f + R_i} v_{in,2} = \frac{R_i}{R_f + R_i} v_o + \frac{R_f}{R_f + R_i} v_{in,1}$$

$$R_f v_{in,2} = R_i v_o + R_f v_{in,1}$$

$$v_o = \frac{R_f}{R_i} \left(v_{in,2} - v_{in,1}\right)$$

This circuit is called a "Differential Amplifier" because it amplifies the difference between the two inputs. The gain can be set using different values for R_f and R_i.

Transfer Functions

In electronics and control systems, it is common to use a **transfer function** to describe the relationship between the input and output. The transfer function is defined as $\frac{v_o}{v_{in}}$.

A generic system with transfer function G is represented with a block diagram as follows

where $v_o = Gv_{in}$

For example, the transfer function of a voltage follower is $G = 1$ and the transfer function of the amplifier is $G = 1 + \frac{R_f}{R_i}$.

A comparator does not have a meaningful transfer function since the ratio of v_o to v_{in} is not constant.

Transfer Functions and the s-domain

For op-amp circuits involving time-dependent components like capacitors and inductors, the output is no longer a multiple of the input, at least not when viewing the signals as functions of time.

The problem is reconciled by working in the s-domain, which is a generalization of the phasor domain studied in 2E04.

When working in the s-domain, the behaviours of capacitors, inductors, and resistors are nicely represented by algebraic expressions in terms of a new s variable.

s is a variable and does not mean "seconds" or any other unit.

Impedances

A component's **impedance** is the ratio of the the voltage across its terminals to the current passing through the components, with these ratios considered in the s-domain, that is $Z = \dfrac{V(s)}{I(s)}$. The units of impedance are ohms.

A resistor with resistance R in ohms has impedance R.

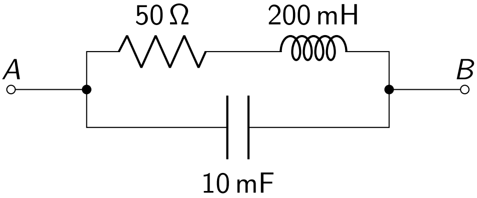
A capacitor with capacitance C in farads has impedance $\dfrac{1}{Cs}$.

An inductor with inductance L in henries has impedance Ls.

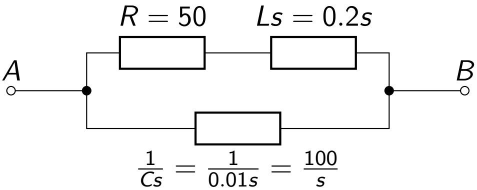
When given a circuit involving capacitors and inductors, the overall impedance can be solved by replacing all components with their respective impedances, treating them like resistors, and using regular circuit analysis techniques.

Impedance Example

What is the overall impedance between point A and point B?



First, replace all components with their impedances.



Impedance Example

The top branch is a serial combination, so take the sum of the impedances of the resistor and inductor: $Z_{top} = 50 + 0.2s$

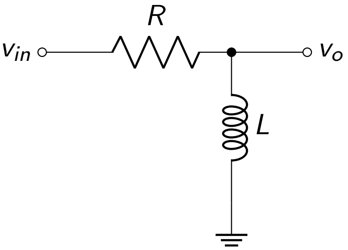
The top and bottom branches are a parallel combination, so use the regular parallel rules to find the total impedance.

$$\begin{aligned}
Z_{AB} &= \left(\frac{1}{Z_{top}} + \frac{1}{Z_{bottom}}\right)^{-1} \\
&= \left(\frac{1}{50 + 0.2s} + \frac{1}{\frac{100}{s}}\right)^{-1} \\
&= \left(\frac{100 + 50s + 0.2s^2}{5000 + 200s}\right)^{-1} \\
&= \frac{200s + 5000}{0.2s^2 + 50s + 100}
\end{aligned}$$

Therefore, the impedance between A and B is $\frac{200s+5000}{0.2s^2+50s+100}$ ohms.

Transfer Function Example

Find the transfer function for the following passive high pass filter.



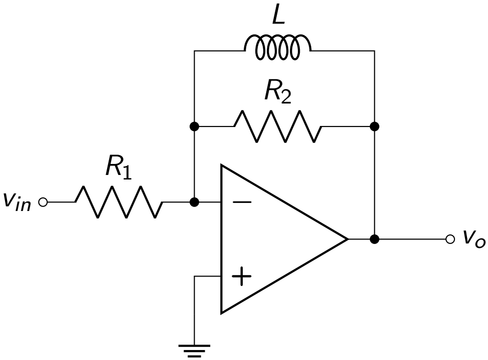
To solve, apply KCL at the v_o node using the impedances of both elements.

$$\frac{v_o - v_{in}}{R} + \frac{v_o - 0}{Ls} = 0$$
$$Lsv_o - Lsv_{in} + Rv_o = 0$$
$$v_o(Ls + R) = Lsv_{in}$$
$$\frac{v_o}{v_{in}} = \frac{Ls}{Ls + R}$$

Therefore, the transfer function is $G(s) = \dfrac{v_o(s)}{v_{in}(s)} = \dfrac{Ls}{Ls + R}$

Op-Amp Transfer Function

The following circuit is an active high pass filter.



Determine its transfer function. Since there is negative feedback, the $v_- = v_+$ assumption can be made. Notice that v_+ is tied to ground, so $v_+ = v_- = 0$.

Op-Amp Transfer Function

Use KCL at v_- and the fact that no current flows into the inverting input to relate v_o to v_{in}.

$$\frac{v_- - v_{in}}{R_1} + \frac{v_- - v_o}{\left(\frac{1}{R_2} + \frac{1}{Ls}\right)^{-1}} = 0$$

Setting $v_- = 0$

$$\begin{aligned}
-\frac{v_{in}}{R_1} - \left(\frac{1}{R_2} + \frac{1}{Ls}\right) v_o &= 0 \\
\frac{Ls + R_2}{R_2 Ls} v_o &= -\frac{v_{in}}{R_1} \\
\frac{v_o}{v_{in}} &= -\frac{R_2 Ls}{R_1 Ls + R_1 R_2} \\
\frac{v_o}{v_{in}} &= -\frac{R_2}{R_1}\frac{Ls}{Ls + R_2}
\end{aligned}$$

Op-Amp Transfer Function

The transfer function of this high pass filter circuit is

$$G(s) = -\frac{R_2}{R_1}\frac{Ls}{Ls + R_2}$$

Notice that by setting R_2 to the same value as R in the passive high-pass filter, then the same $\frac{Ls}{Ls+R}$ term is present in both transfer functions.

The difference with the op-amp filter is that it inverts the signal (due to the negative sign in the transfer function) and its amplitude is scaled by $\frac{R_2}{R_1}$.

Benefits of an Active Filter

This scaling factor highlights one benefit of an active filter: it can add energy to the system. By choosing an $\frac{R_2}{R_1}$ ratio greater than 1, the output signal is an amplified. In contrast, the output amplitude of the passive high pass filter cannot exceed the input amplitude.

In a situation that requires detecting high frequency signals that have a small amplitude, the active filter will provide both functionalities.

Bandwidth

An important input signal is the sinusoid:

$$v_{in}(t) = \sin(\omega t)$$

Most components and circuits will experience some power loss for high frequency inputs.

The **bandwidth** ω_{BW} is defined to be the maximum sinusoid frequency for which the output power is no less than $\frac{1}{2}$ of the input power.

For example, most op-amps have a bandwidth greater than 10 kHz, meaning that frequencies less than this will be processed without "too much" power loss.

Cutoff Frequency

For a filter circuit, the **cutoff frequency** is similar to bandwidth as it represents the first frequency to have its power attenuated by $\frac{1}{2}$. Whether this is a minimum or maximum frequency depends on the type of filter.

Note that for active filters, like the op-amp high pass filter shown above, there may be some amplification factor affecting the output. In this case, the cutoff frequency is typically assumed to be the frequency at which the voltage has dropped by $\frac{1}{\sqrt{2}}$ *relative to the maximum voltage output*.

For example, if the amplification on the active low pass filter is 20, then a 1 V DC signal will be passed to an output of 20 V. The cutoff frequency will be the frequency such that a 1 V sinusoid produces a $\frac{1}{\sqrt{2}} \cdot 20\,\text{V} = 14.14\,\text{V}$ output.

Calculating Bandwidth

Given the transfer function $G(s)$ of a circuit, it is easy to find its bandwidth / cutoff frequency.

1. Since $G(s)$ is a ratio of output to input voltages, set $|G(s)| = \frac{1}{\sqrt{2}}$. (If necessary, accommodate any amplification requirements here, as discussed on the previous slide).
2. Sub in $s = j\omega_{BW}$ where $j = \sqrt{-1}$. This corresponds to an input of $\sin(\omega_{BW} t)$
3. Solve for ω_{BW} using regular complex number arithmetic.

Cutoff Frequency Example

Determine the cutoff frequency for the passive high pass filter shown earlier. Let $R = 2200\,\Omega$ and $L = 200\,\mathrm{mH}$.

Recall the transfer function was $G(s) = \frac{Ls}{Ls+R} = \frac{0.2s}{0.2s+2200}$.

Now follow the steps from the previous slide

$$\begin{aligned}
|G(s)| &= \frac{1}{\sqrt{2}} \\
\left|\frac{0.2j\omega_{BW}}{0.2j\omega_{BW} + 2200}\right| &= \frac{1}{\sqrt{2}} \\
\frac{|0.2j\omega_{BW}|}{|0.2j\omega_{BW} + 2200|} &= \frac{1}{\sqrt{2}} \\
\frac{0.2\omega_{BW}}{\sqrt{(0.2\omega_{BW})^2 + 2200^2}} &= \frac{1}{\sqrt{2}}
\end{aligned}$$

Squaring both sides

$$\frac{(0.2\omega_{BW})^2}{(0.2\omega_{BW})^2 + 2200^2} = \frac{1}{2}$$

Rearranging

$$2 \cdot (0.2\omega_{BW})^2 = (0.2\omega_{BW})^2 + 2200^2$$

$$(0.2)^2{\omega_{BW}}^2 - 2200^2 = 0$$

Using the quadratic formula (or other methods)

$$\omega_{BW} = 11\,\text{kHz}$$

Therefore, the cutoff frequency is 11 kHz. Since this is a high-pass filter, all input frequencies below 11 kHz will produce an output voltage less than $\frac{1}{\sqrt{2}}$ of the input voltage.

The bel unit

When dealing with electronics, a linear gain scale is often too coarse to completely understand a circuit since it compresses all low gains into a small region. By passing to a logarithmic scale, these details are preserved.

The gain of a system is the ratio $\frac{P_{out}}{P_{in}}$. This ratio can alternatively be conveyed by taking its logarithm.

While this log value is technically unitless, we append the unit **bel** [B] to emphasize that this gain is expressed as the log of the power ratio, not the ratio itself.

$$\text{Gain} = \frac{P_{out}}{P_{in}} = \log_{10}\left(\frac{P_{out}}{P_{in}}\right) \text{[B]}$$

The bel unit

Example

A certain filter attenuates an input power by $50\times$. What is the gain as a ratio and in bels?

We are told $P_{out} = \frac{1}{50} P_{in}$, so the gain ratio is $\frac{1}{50}$.
To express in bels, take its log

$$\text{Gain} = \log_{10}\left(\frac{1}{50}\right) = -1.699\,\text{B}$$

The bel unit

Example

An audio amplifier has a gain of 2 B. If the input power is 7 W, what is the output power?

$$\begin{aligned}
\text{Power Gain} &= \log_{10}\left(\frac{P_{out}}{P_{in}}\right) \\
2\,\mathrm{B} &= \log_{10}\left(\frac{P_{out}}{7\,\mathrm{W}}\right) \\
10^2 &= \frac{P_{out}}{7\,\mathrm{W}} \\
700\,\mathrm{W} &= P_{out}
\end{aligned}$$

The output power is 700 W.

The decibel scale

The last two examples show that very large and very small power gains correspond to relatively small values when expressed in bels. For this reason, we typically express gain in **decibels** [dB], which is $\frac{1}{10}$ of a bel.

Given a power gain ratio, its gain in decibels is calculated as

$$\text{Power Gain} = 10\log_{10}\left(\frac{P_{out}}{P_{in}}\right)\,[\text{dB}]$$

For example, $2\,\mathrm{B} = 20\,\mathrm{dB}$ and $-1.699\,\mathrm{B} = -16.99\,\mathrm{dB}$.

The decibel scale

In most cases, we find ourselves working with voltage gain rather than power gain. Using the relationship

$$\frac{P_{out}}{P_{in}} = \left(\frac{V_{out}}{V_{in}}\right)^2$$

and the logarithm laws, we get an expression for power gain in decibels, in terms of voltage gain

$$\text{Gain} = 10\log_{10}\left(\left(\frac{V_{out}}{V_{in}}\right)^2\right)[\text{dB}] = 20\log_{10}\left(\frac{V_{out}}{V_{in}}\right)[\text{dB}]$$

The decibel scale

Example

What is the gain, in decibels, of a filter at its cutoff frequency?

Using the power definition of cutoff, $\frac{P_{out}}{P_{in}} = \frac{1}{2}$:

$$\text{Gain} = 10\log_{10}\left(\frac{P_{out}}{P_{in}}\right) = 10\log_{10}\left(\frac{1}{2}\right) = -3.01\,\text{dB}$$

Using the voltage definition of cutoff, $\frac{V_{out}}{V_{in}} = \frac{1}{\sqrt{2}}$:

$$\text{Gain} = 20\log_{10}\left(\frac{V_{out}}{V_{in}}\right) = 20\log_{10}\left(\frac{1}{\sqrt{2}}\right) = -3.01\,\text{dB}$$

As would be expected, both methods produce the same result.

The decibel scale

Remark

The cutoff frequency is sometimes called the "$-3\,\mathrm{dB}$ frequency" since it corresponds to a gain of roughly $-3\,\mathrm{dB}$.

Additionally, every $-3\,\mathrm{dB}$ change corresponds to another halving of the power gain.

Common Mode Gain

One of the initial assumptions about op-amps was that it amplifies the *difference* between its two inputs, that is

$$v_o = K(v_+ - v_-)$$

An imperfection with real op-amps is common mode gain, which is an effect where some multiple of the *average* value of the two inputs is added to the output signal. This means that the output voltage is actually

$$v_o = K(v_+ - v_-) + K_{cm}\left(\frac{v_+ + v_-}{2}\right)$$

where K_{cm} is the **common mode gain** factor. It may be positive or negative and is typically quite small compared to K, but some scenarios require its consideration.

Common Mode Rejection Ratio

To express how much the op-amp output is affected by common mode gain, we can consider the **common mode rejection ratio** CMRR, defined as

$$CMRR = \frac{K}{|K_{cm}|}$$

This indicates how much greater the differential gain K is compared to the common mode gain K_{cm}. The absolute value on K_{cm} ensures this value is positive.

Example

If $K = 200\,000$ and $K_{cm} = -6.5$, then $CMRR = \frac{200\,000}{|-6.5|} = 30\,769.2$

Common Mode Rejection Ratio

Since the CMRR is typically a large value, and it represents a ratio of voltages (roughly the differential voltage to the common mode voltage), it is often expressed in units of decibels. Use the voltage decibel gain equation to express CMRR as

$$CMRR = 20\log_{10}\left(\frac{K}{|K_{cm}|}\right)\ [\text{dB}]$$