Group Colourings of Knots

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Introduction

The strength of an invariant for distinguishing knots is related to the number of unique values which is can assume. Invariants with a smaller range are inherently weaker. For example, the "tricolourability" invariant only separates knots into 2 classes, and the unknotting number is limited to positive integers, lacking the resolution to distinguish many knots.

A more powerful invariant can be created by extracting a group structure from a knot. Groups can become very complicated, granting us the ability to differentiate more knots by leveraging the extensive theorems from abstract algebra.

The groups can become so big that we will instead study surjective homomorphisms of the knot group onto simpler groups. This mapping can be described by "colouring" (i.e. labelling) the arcs of a knot diagram with group elements according to some relations. This "group colouring" is a generalization of tricolourability and mod p-colourings.

Let's start with a brief construction of the fundamental group of a knot.

1 The Fundamental Group of a Knot

We will define an algebraic group for a knot. As an invariant, properties of this group can be used to study knots, and more importantly differentiate them. This group will consist of *loops* which may or may not "loop around" parts of the knot, and the relations between these loops will define the structure of our group.

1.1 Loops and Homotopies

A **loop** is a continuous function $\gamma : [0, 1] \to \mathbb{R}^3 \setminus K$ which starts and ends at the same point $p = \gamma(0) = \gamma(1)$. Exluding the knot $K \subset \mathbb{R}^3$ from the loop's range means that it must go around the knot without intersecting it. The $+t$ progression induces an orientation for the loop.

Consider two loops with the same basepoint p. If there is a continous deformation from one loop to another, such that every intermediate function is a loop with basepoint p, then we say the loops are **homotopic**. To retain the structure of our knot, it is important that the the deformation's intermediate loops remain in $\mathbb{R}^3 \setminus K$. The **homotopy class** $[\gamma]$ is the set of all loops which are homotopic to a loop γ. Homotopy is clearly an equivalence relation on the set of loops with the same basepoint.

In Figure 1, γ_{1a} and γ_{1b} are in the same homotopy class. The loop γ_2 is not homotopic to the others since it wraps around K twice and in the opposite direction.

Remark. The space $\mathbb{R}^3 \setminus K$ is path connected, so any loop from basepoint p_1 can be expressed as a loop from p_2 by prepending and appending the path from p_2 to p_1. Therefore, we will not concern ourselves with the specific basepoint p and will assume that it has been fixed.

1.2 The Fundamental Group

Two loops with the same basepoint can be concantenated to form a new loop. More formally, define

$$\gamma_1\gamma_2 = \begin{cases} \gamma_1(2t) & 0 \le t \le \frac{1}{2} \\ \gamma_2(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

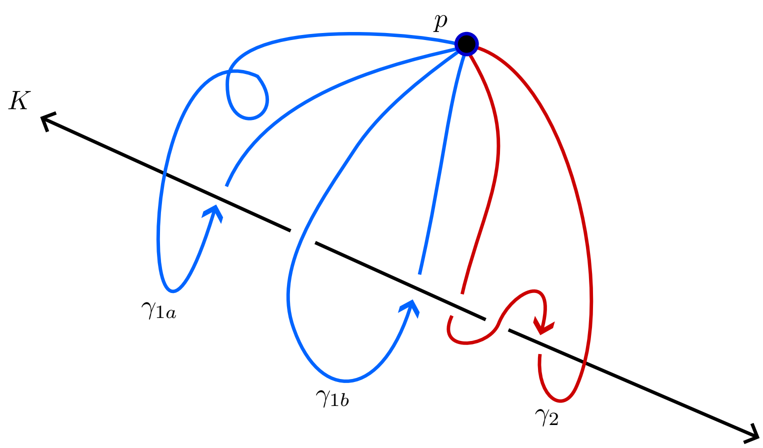

Figure 1: Three loops in $\mathbb{R}^3 \setminus K$ with basepoint p.

This product is a proper loop with the same basepoint as its constituents. We now define a group using this operation.

Definition 1.1. The **fundamental group** $\pi_1(K)$ **of a knot** K consists of the set of homotopy classes of loops in $\mathbb{R}^3 \setminus K$. The group operation is concatenation of loops, the identity is the constant loop $\gamma_e(t) = p$, and the inverse of $\gamma(t)$ is $\bar{\gamma}(t) = \gamma(1-t)$.

The fundamental group satisfies the group axioms and is well defined with respect to the representative loop elements of each homotopy class.

Theorem 1.1. *The fundamental group is a knot invariant.*

Proof. Two knots K_1 and K_2 in $\mathbb{R}^3$ are equivalent if there is a continuous function F taking K_1 to K_2 without causing the knot to pass through itself. By deforming the whole space, this takes a loop $\gamma \in \pi_1(\mathbb{R}^3 \setminus K_1; p)$ to a loop $F(\gamma) \in \pi_1(\mathbb{R}^3 \setminus K_2; F(p))$. ■

This group can be exceedingly complicated, even for simple knots. We can begin to understand it by investigating the relations between loops near crossings. Consider an oriented diagram for a knot K and revert to the 3D knot by "lifting" each over-arc along the z-axis. Pick a point p high above the knot in the $+z$ direction and consider loops from p around the arcs at this crossing.

To each arc x we identify the element of $\pi_1(K)$ given by the class of loops which wrap around x once in the positive direction (point your right thumb along the oriented arc and your fingers will curl in the positive loop direction).

Example. Figure 2 shows a positive cross where the overstrand, incoming and outgoing understrands have been labelled x, y and z respectively. $[\gamma_1]$ is represented by the group element x^{-1} and $[\gamma_2]$ by y^2

Loops around the overstrand are homotopic as they can be "slid over" the crossing with no issue. In contrast, loops around either understrand arc are not generally homotopic as any deformation is blocked by the overstrand.

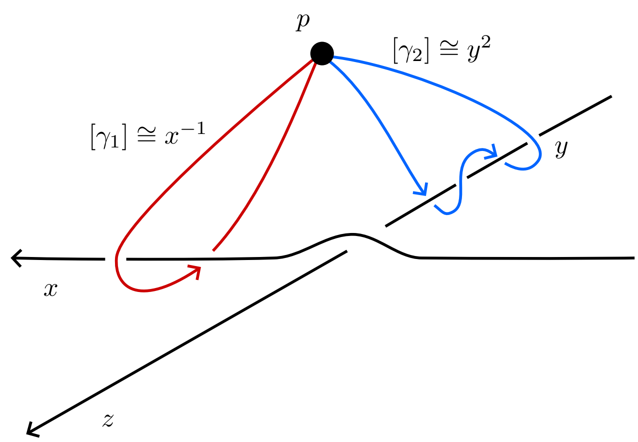

Figure 2

1.3 The Wirtinger Presentation

What relations can we expect between these group elements? If we loop around three consecutive "ends" of the crossing arcs, then the composed loop is homotopic to one around the fourth arc, as shown in Figure 3.

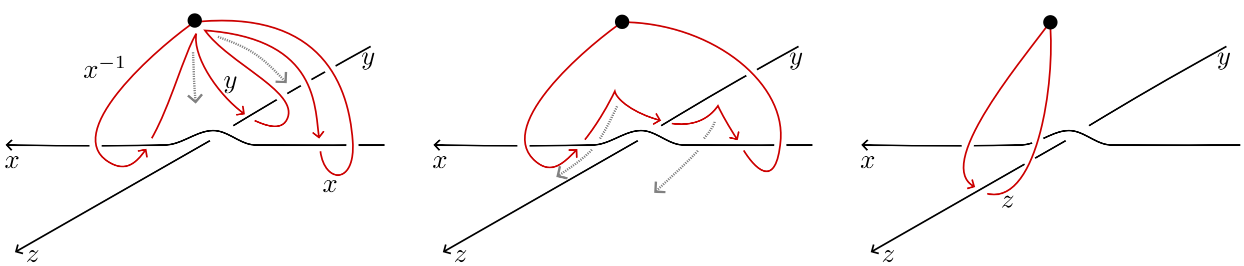

Figure 3: A homotopy from $x^{-1}yx$ to z.

In particular, for a positive crossing labelled as above, we have the relation $x^{-1}yx = z$. The relation is $xyx^{-1} = z$ at a negative crossing. See Figure 4.

So far, we have identified an element of the group $\pi_1(K)$ to each arc in the diagram and have found a relation induced by each crossing. A theorem of Wirtinger asserts that this is a finite presentation of the fundamental group.

Theorem 1.2 (Wirtinger Presentation)**.** *The fundamental group of a knot K in $\mathbb{R}^3$ has a presentation given by*

$$\pi_1(K) = \langle x_1, \ldots, x_g \mid r_1, \ldots, r_n \rangle$$

Where $\{x_i\}$ are the arcs in an oriented diagram for K and each relator r_i is a word in $\{x_i\}$ in the form of Figure 4.

Example. The fundamental group of the right-handed trefoil (Figure 5) has the Wirtinger presentation

$$\pi_1(3_1) = \left\langle x, y, z \mid x^{-1}yx = z, y^{-1}zy = x, z^{-1}xz = y \right\rangle$$

This presentation can be simplified using the third relator to express z in terms of x, y.

$$\begin{aligned}\pi_1(3_1) &= \left\langle x, y \mid y^{-1}(x^{-1}yx)y = x, (x^{-1}yx)^{-1}x(x^{-1}yx) = y \right\rangle \\ &= \langle x, y \mid xyx = yxy \rangle\end{aligned}$$

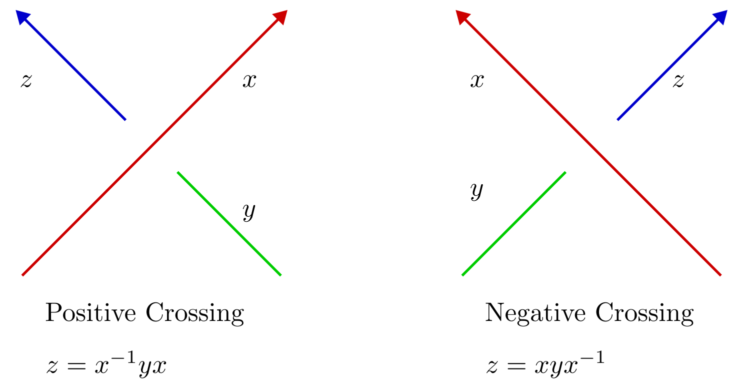

Figure 4: The Wirtinger Relations

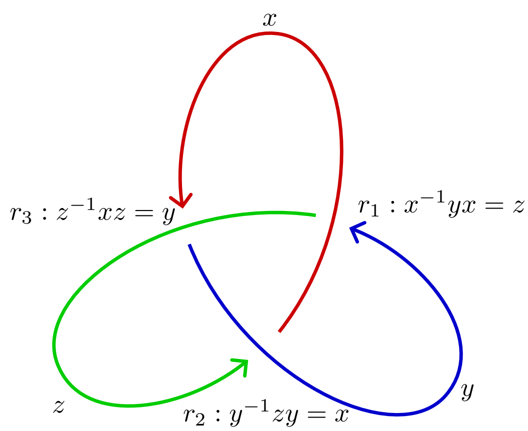

Figure 5: The Wirtinger Presentation of $\pi_1(3_1)$

2 Group Colourings

In Section 1, we constructed the Wirtinger presentation of the fundamental group of a knot. This presentation lets us work with arcs and crossings in a knot diagram rather than loops in 3D space.

The fundamental group of even the simplest knot 3_1 is infinite and non-abelian, and the complexity will only grow for knots with more crossings. We cannot hope to simplify, let alone identify, the fundamental groups of a general knot.

A common theme surrounding knot invariants is the tradeoff between identifying power and computational complexity. Instead of studying the entire group, we will consider other groups which can be used to relabel the arcs of a knot diagram while respecting the Wirtinger relations.

2.1 Consistent Labellings

Definition 2.1. Let K be a knot and G be a group (not necessarily the fundamental group of K). A labelling of the arcs in an oriented diagram for K using elements of G is said to be **consistent** if it satisfies the Figure 4 relations at each crossing.

Example. The trefoil can be labelled consistently with the symmetric group S_3 using the elements $x = (1\ 2)$, $y = (1\ 3)$ and $z = (2\ 3)$.

Example. For any knot K and group G, labelling all arcs with the same element $g \in G$ is consistent since we trivially have $g^{-1}gg = g$ at each crossing.

Theorem 2.1. *If a diagram of K can be consistently labelled with a group G, then any diagram for K can also be labelled consistently with G.*

Proof. We will show that consistent labellings are undisturbed by Reidemeister moves.

The R1 case is simple - labelling both arcs with g after the move is consistent. For the reverse R1 move, the two arcs must already be labelled identically since the relatioin $g^{-1}hg = g$ implies $h = g$.

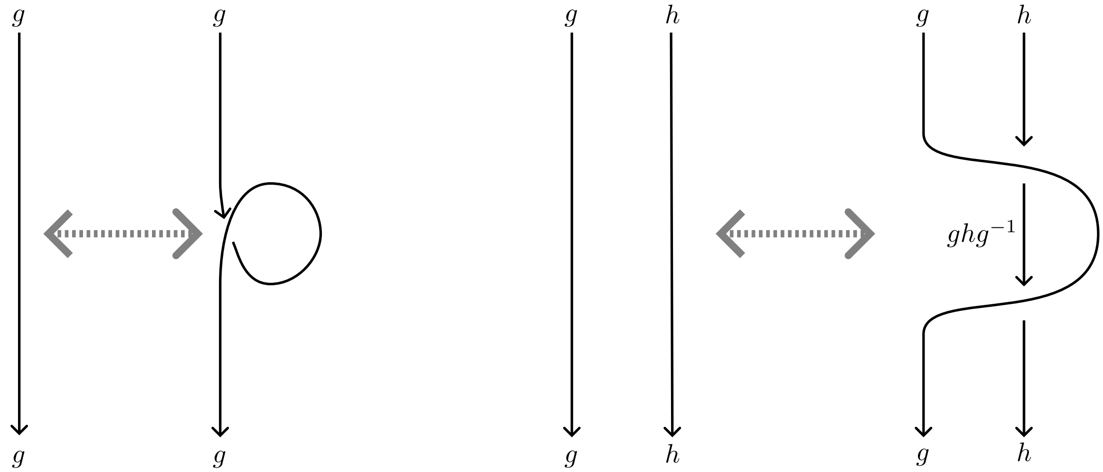

Figure 6: Consistency across R1 and R2 moves.

For the R2 move, if all strands are labelled identically then a similar mono-labelling will hold after the move. It is only interesting when the strands are labelled differently.

To be consistent at the (negative) upper crossing, we must label the middle arc ghg^{-1}. We verify that this is consistent with the (positive) lower crossing:

$$g^{-1}(ghg^{-1})g = (g^{-1}g)h(g^{-1}g) = h$$

Applying this reasoning in reverse shows that it is not possible to consistently label the two h arcs with different group elements, so the reverse R2 proof also follows from this diagram.

The R3 cases are presented in Figure 5.4 of [1]. ■

Lemma 2.2. *If an oriented diagram of K can be labelled consistently with G, then the reverse oriented diagram can also be labelled as such.*

Proof. (Theorem 1 in §5 of [1]) Label the arcs in the reverse diagram with the inverse of each label in the original diagram (Figure 7). This new labelling is consistent since the new crossing relation is algebraically equivalent to the original relation.

$$x^{-1}yx = z \quad \Longleftrightarrow \quad xz^{-1}x^{-1} = y^{-1}$$

Notice that the incoming strand y is the outgoing strand after the reversal. ■

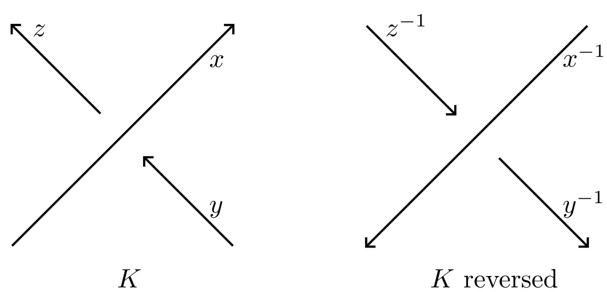

Figure 7

Combining Theorem 2.1 and Theorem 2.2 means that the consistent labellings with some group is an invariant of the *unoriented* knot.

Lemma 2.3. *If a knot is consistently labelled with an abelian group, then all arcs are labelled with the same group element.*

Proof. In an abelian group, the Wirtinger relation collapses to

$$g^{-1}kg = h \implies g^{-1}gk = h \implies k = h$$

Thus, arc labels cannot change over a crossing, so all arcs must be labelled identically. ■

Theorem 2.4. *The labels in a consistent labelling are in the same conjugacy class of G.*

Proof. Index the arcs in the diagram sequentially (i.e. the incoming and outgoing understrands at each crossing are x_i and x_{i+1}, see fig. 12 for such an indexing).

The Wirtinger relations state that $x_{i+1} = y^{-1}x_i y$ where y is the overstrand, so x_{i+1} and x_i are conjugate. Conjugacy is an equivalence relation. In particular, it is transitive.

$$x_1 = y^{-1}x_0 y \quad x_2 = z^{-1}x_1 z \quad \implies \quad x_2 = (yz)^{-1}x_0(yz)$$

A knot has a single strand, so x_i must be conjugate to all other x_j. ■

2.2 Group Colourings

The previous section showed that consistent labellings of a knot with G form an invariant. The immediate question is: Which groups can and should be used to label a knot?

An earlier example demonstrated that a knot K can be consistently labelled using a single element of any group G, but this trivial labelling is a mere algebraic artifact. We wish to study deeper representations of $\pi_1(K)$ by requiring a stronger labelling.

Definition 2.2. For a group G, a **G-colouring** of a knot K is a consistent labelling of K with a generating subset of G.

By requiring that the labels can generate the entire group, a G-colouring represents a true relation between the structure of $\pi_1(K)$ and G.

Example. The trefoil is both $S3$ and $S4$ colourable.

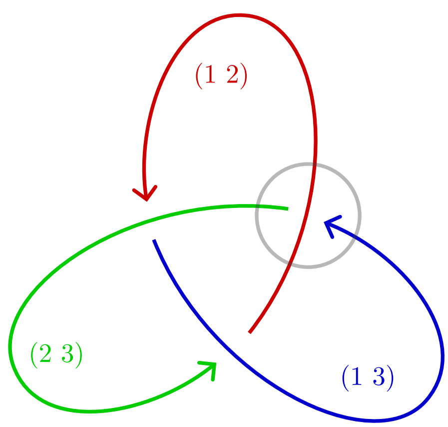

Figure 8: $S3$-colouring of the trefoil.

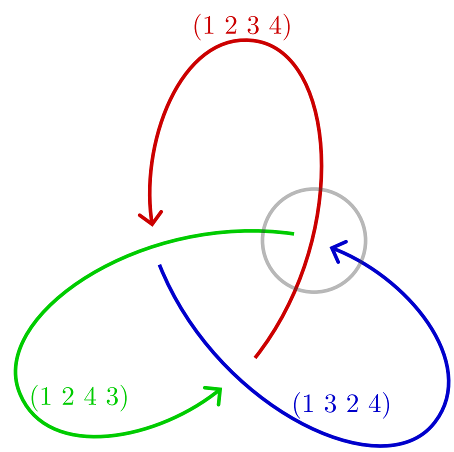

Figure 9: $S4$-colouring of the trefoil.

Let's verify the Wirtinger relation at the circled crossings.

Figure 8	$(1\ 2)^{-1}(1\ 3)(1\ 2) = (1\ 2\ 3)(1\ 2) = (2\ 3)$
Figure 9	$(1\ 2\ 3\ 4)^{-1}(1\ 3\ 2\ 4)(1\ 2\ 3\ 4) = (1)(2\ 3\ 4)(1\ 2\ 3\ 4) = (1\ 2\ 4\ 3)$

While Figure 8 is also a consistent labelling with elements of $S4$, the three elements do not generate the whole group, so a different labelling is needed to show that the knot is $S4$-colourable.

Remark. $S3$ and $S4$ are special: a knot is $S3$-colourable if and only if it is $S4$-colourable. See [4].

Theorem 2.5. *A G-colouring for a knot K defines a surjective homomorphism between $\pi_1(K)$ and G.*

Proof. Given a G-colouring for K, define a map $\varphi : \pi_1(K) \to G$ which takes each generator x_i to its label g_i. Extend this map over the rest of $\pi_1(K)$ by mapping any word in the generators $\{x_i\}$ to a word in the corresponding labels $\{g_i\}$.

$$\varphi(x_i x_j x_k) = \varphi(x_i)\varphi(x_j)\varphi(x_k)$$

The labels $\{g_i\}$ generate G, so for all $g \in G$, we can write $g = g_{n_1} g_{n_2} \cdots g_{n_k}$ where g_{n_i} are labels in the colouring. The definition of φ provides (at least) one element $x_{n_i} \mapsto g_{n_i}$, so

$$\varphi(x_{n_1} x_{n_2} \cdots x_{n_k}) = g_{n_1} g_{n_2} \cdots g_{n_k} = g$$

This is true for all $g \in G$, so the map is surjective.

The map φ is a homomorphism because labels in a G-colouring must satisfy all of the same Wirtinger relations which define $\pi_1(K)$ ■

Theorem 2.6. *The unknot is G-colourable if and only if G is cyclic.*

Proof. Assume G is cyclic. Given a diagram for the unknot, label every arc with a generator of G. This is a consistent labelling which clearly generates G.

Suppose the unknot is G-colourable. Then there is a surjective homomorphism $\varphi : \pi_1(U) \to G$. The fundamental group of the unknot is $\mathbb{Z}$ which is abelian, so the image of φ must also be abelian. Thus, by Theorem 2.3, all arcs are labelled with the same element $g \in G$. This single element must generate G so the group is cyclic. ■

2.3 Fox-n Colourings

The construction of G-colourings is very similar to that of mod-p colourings.

Definition 2.3. For p prime, a p-colouring of a knot K is a labelling of the arcs using the integers $\{0, 1, \ldots, p-1\}$, subject to the crossing relation

$$2x - y - z \equiv 0 \mod p$$

Where x is the label on the overstrand and y,z are the labels on the understrands.

A knot is **p-colourable** if it has a non-constant p-colouring.

This equation is similar to the Wirtinger relation $gkg^{-1} = h$ so we may be tempted to say that a p-colouring is a group colouring with the integers mod p, but this is not the case. $\mathbb{Z}/p\mathbb{Z}$ is an abelian group, so by Theorem 2.3, there are only trivial group colourings, but non-trivial mod-p colourings exist (ex. Figure 10).

To describe p-colourings as a group colouring, we need a group with additivity properties similar to $\mathbb{Z}/p\mathbb{Z}$ but that is non-abelian. Enter the dihedral groups.

Definition 2.4. The **dihedral group** D_n, describes the symmetries of a regular n-sided polygon. It has two generators: s describing a flip and r for a $\frac{360}{n}^\circ$ rotation. The group has a presentation

$$D_n = \langle r, s \mid r^n = s^2 = 1, sr = r^{-1}s \rangle$$

The presentation means that any $g \in D_p$ can be written as either $g = r^i$ or $g = sr^i$ for some $i \in \{0, 1, \ldots p-1\}$.

The r element generates a subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Theorem 2.7. *Mod-p colourings are in 1-to-1 correspondence with D_p group colourings.*

Proof. Start with a p-colouring for K. Define a function $\varphi : \mathbb{Z}/p\mathbb{Z} \to D_p$ by $\varphi(i) = sr^i$ and verify that this satisfies the Wirtinger relations at each crossing.

Let x, y, and z be the overstrand and two understrand labels in $\mathbb{Z}/p\mathbb{Z}$. Orient the diagram such that y is the outgoing understrand, and WLOG suppose the crossing is positive. The D_p Wirtinger relation is satisfied if

$$\varphi(x)\varphi(y)\varphi(x)^{-1} = \varphi(z)$$

or equivalently

$$\varphi(x)\varphi(y)\varphi(x)^{-1}\varphi(z)^{-1} = 1$$

Evaluate φ and apply the dihedral group relations ($sr^k = r^{-k}s$, $s^{-1} = s$, etc.) to rearrange the group elements.

$$\begin{aligned}
\varphi(x)\varphi(y)\varphi(x)^{-1}\varphi(z)^{-1} &= sr^x sr^y (sr^x)^{-1}(sr^z)^{-1} \\
&= sr^x sr^y r^{-x} sr^{-z} s \\
&= sr^x sr^{-x+y} r^z ss \\
&= sr^x sr^{-x+y+z} \\
&= sr^x r^{x-y-z} s \\
&= sr^{2x-y-z} s
\end{aligned}$$

The labels came from a p-colouring, so $2x - y - z \equiv 0 \mod p \implies 2x - y - z = kp$ for some $k \in \mathbb{Z}$.

$$sr^{2x-y-z}s = s(r^p)^k s = s1^k s = s^2 = 1$$

So the Wirtinger relation is satisfied and $\varphi(i)$ gives a D_p-colouring.

We require a lemma to prove the opposite direction.

Lemma 2.8. *The conjugacy class of s in D_p is the elements of the form sr^i.*

Proof. First show sr^k is conjugate to s for all k. Let x be a solution to $2x + k \equiv 0 \mod p$ which exists since p is prime. Conjugate sr^k by r^x.

$$(r^x)^{-1} sr^k r^x = r^{-x} sr^{k+x} = sr^x r^{k+x} = sr^{2x+k} = sr^{xp} = s$$

Now show that r^k is not in the class with s by conjugating it with all other elements $g \in D_p$. If $g = r^i$, then

$$(r^i)^{-1} r^k r^i = r^{-i+k+i} = r^k \neq s$$

and if $g = sr^i$ then

$$(sr^i)^{-1} r^k sr^i = r^{-i} ssr^{-k} r^i = r^{-k} \neq s$$

■

Suppose we have a D_p-colouring for K and wish to construct a mod-p colouring. The labels must generate D_p, so at least one label must contain the s generator. Let $g = sr^i$ be this label. By Theorem 2.4, all labels are conjugate to g, so the lemma guarantees all labels are of the form sr^k. Define $\bar{\varphi} : \left\{ sr^k \mid k \in \mathbb{Z}/p\mathbb{Z} \right\} \to \mathbb{Z}/p\mathbb{Z}$ by $\bar{\varphi} : sr^k \mapsto k$. This is the inverse of the φ map from earlier in the proof, so a similar argument shows that the mod-p colouring obtained by applying $\bar{\varphi}$ to the D_5 labels satisfies the $2x - y - z \equiv 0 \mod p$ relation.

Finally, D_p is not cyclic, so there must be at least two different labels sr^x and sr^y, so the the mod-p labelling from $\bar{\varphi}$ is non-constant. ■

Example. The next two figures show corresponding mod-5 and D_5 colourings of the figure-8 knot.

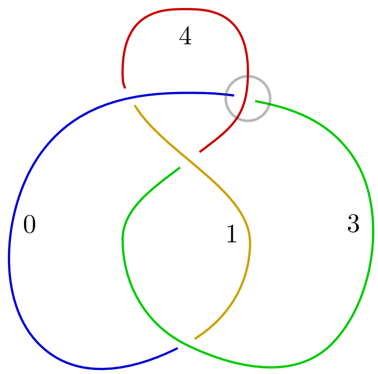

Figure 10: A mod-5 colouring of 4_1.

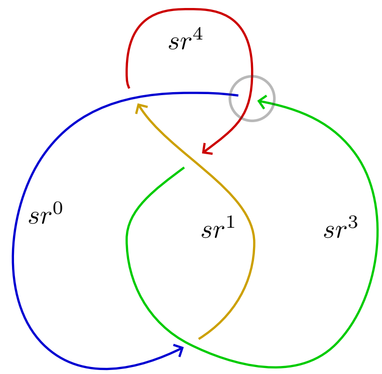

Figure 11: A D_5-colouring of 4_1.

We can verify that the circled crossings satisfy their respective relations.

$$\begin{aligned}
&\text{Figure 10} \quad 2(4) - 3 - 0 = 5 \equiv 0 \mod 5 \\
&\text{Figure 11} \quad sr^4 sr^3 (sr^4)^{-1} \\
&\qquad\qquad = sr^4 r^{-3} s r^{-4} s \\
&\qquad\qquad = sr^{4-3} r^4 ss \\
&\qquad\qquad = sr^{4+4-3} \\
&\qquad\qquad = sr^5 = sr^0
\end{aligned}$$

3 An Application of Group Colourings

Section 2.3 showed that group colourings are a generalization of mod-p colourings. This closing section will show that group colourings are indeed more powerful than their modular cousin.

The $(5, 3)$ torus knot is shown in Figure 12.

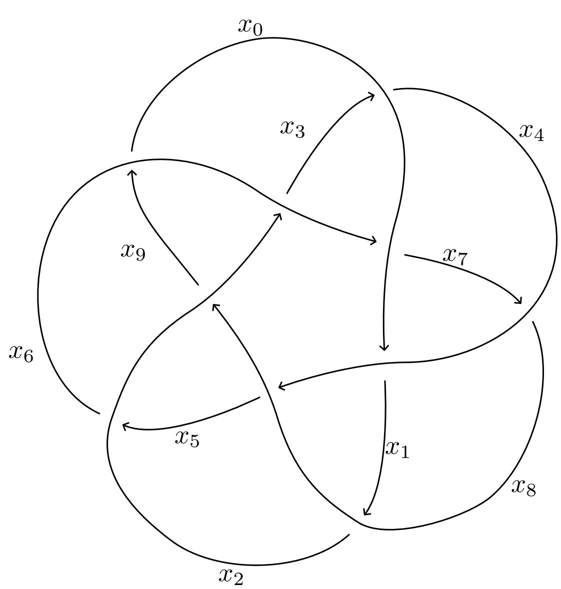

Figure 12: $T_{5,3}$

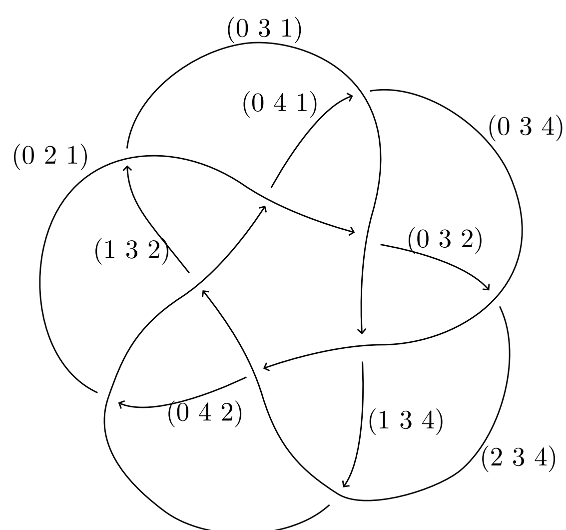

Figure 13: An $A5$-colouring of $T_{5,3}$

The determinant of $T_{5,3}$ is 1 (see 10_{124} at [2]). By Theorem 4 in §3 of [1], it is not mod-p colourable (equivalently, not D_p-colourable) for any prime p. Therefore, these simple colourings are unable to distinguish

$T_{5,3}$ from the unknot.

However, expanding beyond dihedral groups lets us analyze the structure of $\pi_1(T_{5,3})$ more freely. In particular, Figure 13 shows a labelling of the knot by the alternating group on five letters. This group is not cyclic, so we can distinguish $T_{5,3}$ from the unknot by Theorem 2.6.

Example. We can verify the colouring at the upper-right crossing. This crossing is positive and has x_0 as the overstrand.

$$x_0^{-1} x_3 x_0 = (0\ 3\ 1)^{-1}(0\ 4\ 1)(0\ 3\ 1) = (1\ 3\ 0)(0\ 4\ 1)(0\ 3\ 1) = (0\ 3\ 4)(1) = x_4$$

Conjugating the understand x_3 by the overstrand gives x_4, so the labelling is consistent at this crossing.

This labelling was computed by a brute-force Python script using the `SymPy` library [3]. A labelling for $T_{5,3}$ is completely determined by the labels on x_0, x_4, and x_8, so the script "guesses" all possible labels for these arcs. Theorem 2.4 is applied by limiting the 3 guesses to a single conjugacy class. This improves execution time without missing any solutions.

The Wirtinger relations determine the remaining 7 arcs. If the labelling is non-trivial and consistent, then the colouring is saved as a successful solution. See Appendix A for the source code.

References

[1] C. Livingston. *Knot Theory*. Number v. 24 in Carus mathematical monographs. Mathematical Association of America, 1993.

[2] Charles Livingston and Allison H. Moore. Knotinfo: Table of knot invariants. URL: https://knotinfo.math.indiana.edu, December 2024.

[3] Aaron Meurer, Christopher P. Smith, Mateusz Paprocki, Ondřej Čertík, Sergey B. Kirpichev, Matthew Rocklin, AMiT Kumar, Sergiu Ivanov, Jason K. Moore, Sartaj Singh, Thilina Rathnayake, Sean Vig, Brian E. Granger, Richard P. Muller, Francesco Bonazzi, Harsh Gupta, Shivam Vats, Fredrik Johansson, Fabian Pedregosa, Matthew J. Curry, Andy R. Terrel, Štěpán Roučka, Ashutosh Saboo, Isuru Fernando, Sumith Kulal, Robert Cimrman, and Anthony Scopatz. Sympy: symbolic computing in python. *PeerJ Computer Science*, 3:e103, January 2017.

[4] Kenneth A. Perko. *OCTAHEDRAL KNOT COVERS*, pages 47–50. Princeton University Press, Princeton, 1975.

A Python code for $A5$ colouring of $T_{5,3}$

Click for a live demo

```python
# Blake Freer, 2024
# https://github.com/BlakeFreer

from itertools import product

from sympy.combinatorics import Permutation, PermutationGroup
from sympy.combinatorics.named_groups import AlternatingGroup

GROUP = AlternatingGroup(5)


def try_solve(x0, x4, x8):
    x1 = (~x4) * x0 * x4
    x5 = (~x8) * x4 * x8
    x2 = (~x8) * x1 * x8
    x9 = (~x2) * x8 * x2
    x6 = (~x2) * x5 * x2
    x3 = (~x6) * x2 * x6
    x7 = (~x0) * x6 * x0

    assert x0 == (~x6) * x9 * x6, "Inconsistent at x0"
    assert x4 == (~x0) * x3 * x0, "Inconsistent at x4"
    assert x8 == (~x4) * x7 * x4, "Inconsistent at x8"

    solution = [x0, x1, x2, x3, x4, x5, x6, x7, x8, x9]

    assert PermutationGroup(*solution).equals(GROUP), "Doesn't generate."

    return solution


def solve_conj_class(conj: set[Permutation]):
    solutions = []
    for x0, x4, x8 in product(conj, repeat=3):
        try:
            solutions.append(try_solve(x0, x4, x8))
        except AssertionError:
            pass
    return solutions


for cls in GROUP.conjugacy_classes():
    solutions = solve_conj_class(cls)
    print(
        f"Found {len(solutions)} colourings with the conjugacy class of {list(cls)[0]}."
    )

    if solutions:
        print("One example is:")
        for idx, p in enumerate(solutions[0]):
            print(f"x{idx} = {p}")
```

A.1 Output

```
$ python t53.py
Found 0 colourings with the conjugacy class of (4).
Found 120 colourings with the conjugacy class of (2 4 3).
One example is:
x0 = (2 4 3)
x1 = (0 3 4)
x2 = (4)(0 1 3)
x3 = (4)(0 3 2)
x4 = (0 2 4)
x5 = (4)(0 1 2)
x6 = (4)(1 3 2)
x7 = (1 2 4)
x8 = (0 1 4)
x9 = (1 3 4)
Found 120 colourings with the conjugacy class of (0 4)(2 3).
One example is:
x0 = (0 4)(2 3)
x1 = (4)(0 2)(1 3)
x2 = (1 4)(2 3)
x3 = (0 1)(2 4)
x4 = (0 3)(1 4)
x5 = (0 2)(3 4)
x6 = (4)(0 3)(1 2)
x7 = (1 3)(2 4)
x8 = (0 4)(1 2)
x9 = (0 1)(3 4)
Found 120 colourings with the conjugacy class of (0 4 3 1 2).
One example is:
x0 = (0 4 3 1 2)
x1 = (0 1 3 2 4)
x2 = (0 3 4 2 1)
x3 = (0 4 1 2 3)
x4 = (0 1 4 3 2)
x5 = (0 2 1 3 4)
x6 = (0 4 2 3 1)
x7 = (0 1 2 4 3)
x8 = (0 3 2 1 4)
x9 = (0 2 3 4 1)
Found 120 colourings with the conjugacy class of (0 1 2 3 4).
One example is:
x0 = (0 1 2 3 4)
x1 = (0 4 1 3 2)
x2 = (0 3 1 2 4)
x3 = (0 4 1 3 2)
x4 = (0 2 4 3 1)
x5 = (0 4 1 3 2)
x6 = (0 2 1 4 3)
x7 = (0 4 1 3 2)
x8 = (0 1 4 2 3)
x9 = (0 4 1 3 2)
```